

L -functions of symmetric powers of the generalized Airy family of exponential sums

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Abstract

This paper looks at the L -function of the k -th symmetric power of the $\overline{\mathbb{Q}}_\ell$ -sheaf Ai_f over the affine line $\mathbb{A}_{\mathbb{F}_q}^1$ associated to the generalized Airy family of exponential sums. Using ℓ -adic techniques, we compute the degree of this rational function and local factors at infinity.

1 Introduction

In this paper we study the L -function attached to the k -th symmetric power of the $\overline{\mathbb{Q}}_\ell$ -sheaf Ai_f associated to the generalized Airy family of exponential sums. Symmetric powers appear in the proofs of many arithmetic problems. For instance, Deligne's proof [5] of the Ramanujan-Petersson conjecture relies on the construction of a Galois module coming from the k -th symmetric power of a certain ℓ -adic sheaf. The Sato-Tate conjecture [4] [14] [25] relies on the analytic continuation of the L -function attached to the k -th symmetric power of an ℓ -adic representation coming from an elliptic curve. Another equidistribution result concerning Kloosterman angles was proven by Adolphson [3] using results of Robba's [21] on the L -function of the k -th symmetric power of the ℓ -adic Kloosterman sheaf Kl_2 . Symmetric powers also arise in the proof of Dwork's conjecture [27] [28] [29]. To begin, let us recall the general setup of an L -function of an ℓ -adic representation.

Let \mathbb{F}_q be the finite field of q elements and characteristic p . Let Y be a smooth, geometrically connected, open variety defined over \mathbb{F}_q ; for instance, take Y to be affine s -space $\mathbb{A}_{\mathbb{F}_q}^s$ or the torus \mathbb{G}_m^s . Denote its function field by K , and its corresponding absolute Galois group by $G_K := \text{Gal}(K^{\text{sep}}/K)$. Let V be a finite dimensional vector space over a finite extension field of \mathbb{Q}_ℓ , where $\ell \neq p$. Let $\rho : G_K \rightarrow GL(V)$ be a continuous ℓ -adic representation unramified on Y , and let \mathcal{F} be the corresponding lisse sheaf on Y . Define the L -function of ρ on Y by

$$L(Y, \rho, T) := \prod_{x \in |Y|} \frac{1}{\det(1 - \rho(\text{Frob}_x) T^{\deg(x)})}. \quad (1)$$

By the Lefschetz trace formula, this is a rational function whose zeros and poles may be described using étale cohomology with compact support:

$$L(Y, \rho, T) = \prod_{i=0}^{2\dim(Y)} \det(1 - \text{Frob}_q T | H_c^i(Y \otimes \overline{\mathbb{F}}_q, \mathcal{F}))^{(-1)^{i+1}}$$

Given such a representation, we may construct new L -functions via operations such as tensor, symmetric, or exterior products. Natural questions about these new L -functions concern the determination of their degrees (Euler characteristic) and describing various properties about their zeros and poles. In this paper, we will focus on the symmetric powers of a particular family of exponential sums called the *generalized Airy family*. Other families whose symmetric powers have been investigated are the Legendre family of elliptic curves [2] [8] and the hyperKloosterman family [10] [11] [21]. We note that the former seems to have been motivated by Dwork's p -adic interest in the Ramanujan-Petersson conjecture.

The generalized Airy family is defined as follows. Let f be a polynomial over \mathbb{F}_q of degree d with $p \nmid d$. Let ψ be a nontrivial additive character on \mathbb{F}_q . For each $\bar{t} \in \overline{\mathbb{F}}_q$ define its degree by $\deg(\bar{t}) := [\mathbb{F}_q(\bar{t}) : \mathbb{F}_q]$. It is well-known that the associated L -function of the sequence of exponential sums

$$S_m(\bar{t}) := \sum_{x \in \mathbb{F}_{q^m \deg(\bar{t})}} \psi \circ \text{Tr}_{\mathbb{F}_{q^m \deg(\bar{t})}/\mathbb{F}_q}(f(x) + \bar{t}x) \quad \text{for } m = 1, 2, 3, \dots$$

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is a polynomial of degree $d - 1$:

$$L(f, \mathbb{A}^1, \bar{t}; T) := \exp \left(\sum_{m=1}^{\infty} S_m(\bar{t}) \frac{T^m}{m} \right) = (1 - \pi_1(\bar{t})T) \cdots (1 - \pi_{d-1}(\bar{t})T).$$

As we will describe later, the relative cohomology of this family may be represented ℓ -adically as a lisse sheaf of rank $d - 1$ over \mathbb{A}^1 via Fourier transform. Let us denote this sheaf by Ai_f . The L -function of the k -th symmetric power of Ai_f takes the form:

$$M_k(f, T) := L(\mathbb{A}^1, \text{Sym}^k(\text{Ai}_f), T) := \prod_{t \in |\mathbb{A}^1|} \prod_{a_1 + \cdots + a_{d-1} = k} (1 - \pi_1(t)^{a_1} \cdots \pi_{d-1}(t)^{a_{d-1}} T^{\deg(t)})^{-1},$$

where $|\mathbb{A}^1|$ denotes the set of closed points on \mathbb{A}^1 . By the Lefschetz trace formula, $M_k(f, T)$ is a rational function. The ℓ -adic sheaf Ai_f was extensively studied by N. Katz in [16], where its monodromy group is determined and, as a consequence, an equidistribution result is obtained for the exponential sums in the family ([16, Corollary 20]). From these results it follows that, for $p > 2d - 1$, $M_k(f, T)$ is in fact a polynomial. For $d = 3$, a study of the monodromy group may be avoided using Adolphson's method [3].

Our first main result is the computation of the degree of $M_k(f, T)$ for $p > d$. The degree of the rational function $M_k(f, T)$ equals the k -th coefficient of a generating series which is explicitly given in Corollary 3.4. Simplified formulas are given in section 5 for some particularly nice values of f and p .

As an example of this theorem, consider the family generated by $f(x) = x^d$. Then the degree of $M_k(x^d, T)$ may be described as follows. Let ζ be a primitive $(d-1)$ -th root of unity in $\overline{\mathbb{F}}_q$. Denote by $N_{d-1,k}$ the number of $(d-1)$ -tuples $(a_0, a_1, \dots, a_{d-2})$ of nonnegative integers such that $a_0 + a_1 + \cdots + a_{d-2} = k$ and $a_0 + a_1\zeta + \cdots + a_{d-2}\zeta^{d-2} = 0$ in $\overline{\mathbb{F}}_q$.

Theorem 1.1. *With the notation defined above, we have*

$$\deg M_k(x^d, T) = \frac{1}{d-1} \left[\binom{k+d-2}{d-2} - dN_{d-1,k} \right].$$

It was conjectured in [13] that $M_k(x^3, T)$ is a polynomial for all $p > 3$ since it was shown, in that paper, that $M_k(x^3, T)$ is a polynomial for every odd integer k , and also for every k even with $k < 2p$. Surprisingly, for $p = 5$, $M_k(x^3, T)$ is *not* a polynomial for infinitely many k . This was communicated to the first author by N. Katz and is a consequence of the geometric monodromy group of Ai_{x^3} being finite.

Theorem 1.2. *Suppose $p > 2d - 1$. Then $M_k(f, T)$ is a polynomial which may be factored into a product $Q_k(f, T)P_k(f, T)$, where $P_k(f, T)$ satisfies the functional equation*

$$P_k(f, T) = cT^{\deg(P_k)} \overline{P_k(f, 1/q^{k+1}T)} \quad \text{with } |c| = q^{\deg(P_k)(k+1)/2}$$

and $Q_k(f, T)$ has reciprocal roots of weight $\leq k$. Furthermore, writing $f(x) = \sum_{i=0}^d c_i x^i$, if we assume \mathbb{F}_q contains the $2(d-1)$ -th roots of $-dc_d$ then an explicit description of $Q_k(f, T)$ may be given; see Corollary 4.3.

Describing the p -adic behavior of the reciprocal roots of $M_k(f, T)$ is also of interest. Motivation for such a study comes from Wan's reciprocity theorem [26] of the Gouvêa-Mazur conjecture [12] on the slopes of modular forms; see [2] for the connection between symmetric powers of the Legendre crystal with Hecke polynomials. At present we are able to prove the following improvement to [13]. Assume $p \geq 7$, k is odd and $k < p$. Write $M_k(x^3, T) = 1 + c_1 T + \cdots + c_{(k+1)/2} T^{(k+1)/2}$. Then the q -adic Newton polygon lies on or above the quadratic function $\frac{1}{3}(m^2 + m + km)$ for $m = 0, 1, 2, \dots, \frac{k+1}{2}$. Furthermore, as a consequence of the functional equation, the endpoints of the q -adic Newton polygon of $M_k(x^3, T)$ coincide with this lower bound. We will prove this in a separate paper.

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2 Cohomological interpretation of $M_k(f, T)$

In this section we will study the generalized Airy family of exponential sums from the point of view of ℓ -adic cohomology. We will do so by studying the sheaf Ai_f that represents this family on the affine line \mathbb{A}^1 over the given finite field \mathbb{F}_q . We begin by observing that the map $\mathbb{F}_q \rightarrow \mathbb{C}$ given by $t \mapsto \sum_{x \in \mathbb{F}_q} \psi(f(x) + tx)$ is the Fourier transform with respect to ψ , in the classical sense, of the map $t \mapsto \psi(f(t))$. This will translate, in the cohomological sense, to the fact that Ai_f is the Fourier transform, in the sheaf-theoretical sense, of the $\overline{\mathbb{Q}}_\ell$ -sheaf

that represents the latter map, which is just the pull-back of the Artin-Schreier sheaf associated to ψ via the map given by f . Let us be more precise.

The polynomial f naturally defines a morphism, also denoted by $f : \mathbb{A}_{\mathbb{F}_q}^1 \rightarrow \mathbb{A}_{\mathbb{F}_q}^1$. Let \mathcal{L}_ψ be the Artin-Schreier sheaf on $\mathbb{A}_{\mathbb{F}_q}^1$ associated to ψ (cf. [6, 1.7]). For every finite extension \mathbb{F}_{q^m} of \mathbb{F}_q , every $t \in \mathbb{A}^1(\mathbb{F}_{q^m}) = \mathbb{F}_{q^m}$ and every geometric point \bar{t} over t , we have $\text{Trace}(\text{Frob}_t | \mathcal{L}_{\psi, \bar{t}}) = \psi(\text{Trace}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(t))$, where Frob_t denotes a geometric Frobenius element at t . Consider the pullback $\mathcal{L}_{\psi(f)} := f^* \mathcal{L}_\psi$.

By [16, Theorem 17], for $d \geq 2$ the Fourier transform with respect to ψ of $\mathcal{L}_{\psi(f)}$ (which, in principle, is an element of the derived category $\mathcal{D}_c^b(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell)$) is in fact a (shifted) lisse sheaf on \mathbb{A}^1 , of rank $d-1$ and with $d/(d-1)$ as its single slope at infinity. Its Swan conductor is therefore d . Let us denote this sheaf by $\text{Ai}_f = \mathbf{R}^1 \pi_{t!} \mathcal{L}_{\psi(f(x)+tx)}$, where $\pi_t : \mathbb{A}^2 \rightarrow \mathbb{A}^1$ is the projection $(x, t) \mapsto t$. For every finite extension \mathbb{F}_{q^m} of \mathbb{F}_q , every $t \in \mathbb{F}_{q^m}$ and every geometric point \bar{t} over t we have, denoting $\psi_m = \psi \circ \text{Trace}_{\mathbb{F}_{q^m}/\mathbb{F}_q}$:

$$\text{Trace}(\text{Frob}_t | (\text{Ai}_f)_{\bar{t}}) = - \sum_{x \in \mathbb{F}_{q^m}} \psi_m(f(x) + tx).$$

The characteristic polynomial of the action of a geometric Frobenius element Frob_t at t on the stalk of Ai_f at a geometric point over t has the form

$$L(\text{Ai}_f, t, T) = (1 - \pi_1(t)T) \cdots (1 - \pi_{d-1}(t)T)$$

where $\pi_i(t)$ is a Weil algebraic number of weight 1 (i.e. all its complex conjugates have absolute value $q^{1/2}$) and $\sum_{x \in \mathbb{F}_{q^m}} \psi_m(f(x) + tx) = - \sum_i \pi_i(t)^m$ for all $m \geq 1$. Its k -th “symmetric power” is given by

$$L(k; \text{Ai}_f, t, T) := \prod_{a_1 + \cdots + a_{d-1} = k} (1 - \pi_1(t)^{a_1} \cdots \pi_{d-1}(t)^{a_{d-1}} T).$$

These are the local factors of the L -function of the k -th symmetric power of Ai_f , which is given by the infinite product

$$M_k(f, T) := \prod_{t \in |\mathbb{A}^1|} L(k; \text{Ai}_f, t, T^{\deg(t)})^{-1}$$

The Lefschetz trace formula demonstrates that the zeros and poles of $M_k(f, T)$ may be described in terms of cohomology:

$$M_k(f, T) = \prod_{i=0}^2 \det(1 - \text{Frob } T | H_c^i(\mathbb{A}_{\mathbb{F}_q}^1, \text{Sym}^k \text{Ai}_f))^{(-1)^{i+1}}.$$

Since $\text{Sym}^k \text{Ai}_f$ is a lisse sheaf on the affine line, we have $H_c^0(\mathbb{A}_{\mathbb{F}_q}^1, \text{Sym}^k \text{Ai}_f) = 0$, and the previous formula simplifies to

$$M_k(f, T) = \frac{\det(1 - \text{Frob } T | H_c^1(\mathbb{A}_{\mathbb{F}_q}^1, \text{Sym}^k \text{Ai}_f))}{\det(1 - \text{Frob } T | H_c^2(\mathbb{A}_{\mathbb{F}_q}^1, \text{Sym}^k \text{Ai}_f))}.$$

On the other hand, $H_c^2(\mathbb{A}_{\mathbb{F}_q}^1, \text{Sym}^k \text{Ai}_f)$ is just the space of co-invariants of the sheaf $\text{Sym}^k \text{Ai}_f$, regarded as a representation of the fundamental group $\pi_1(\mathbb{A}_{\mathbb{F}_q}^1)$, which is the k -th symmetric power of Ai_f regarded as a representation of the same group. This is the same as the space of co-invariants for its monodromy group, which is defined to be the Zariski closure of its image in the group of automorphisms of the generic stalk of Ai_f , isomorphic to $\text{GL}(d-1) := \text{GL}(d-1, \overline{\mathbb{Q}}_\ell)$. By [16, Theorem 19], for $p > 2d-1$ the geometric monodromy group of Ai_f is either $\text{SL}(d-1)$ for d even, or $\text{Sp}(d-1)$ for d odd if $c_{d-1} = 0$ and $\mu_p \cdot \text{SL}(d-1)$ for d even or $\mu_p \cdot \text{Sp}(d-1)$ for d odd if $c_{d-1} \neq 0$ (where $f(x) = \sum_{i=0}^d c_i x^i$). In either case, its k -th symmetric power is still an irreducible representation of rank $\binom{d+k-2}{d-2}$ of the monodromy group (because it is an irreducible representation of its subgroup $\text{SL}(d-1)$ or $\text{Sp}(d-1)$), and in particular the space of co-invariants vanishes. More generally, it was proven by O. Šuch ([24, Proposition 1.6]) that, for $p > 2$, either Ai_f has finite monodromy or its monodromy group contains $\text{SL}(d-1)$ or $\text{Sp}(d-1)$. In order to rule out the finite monodromy case for $p \leq 2d-1$ one may use for instance [18, Proposition 8.14.3], which implies that Ai_f has finite monodromy if and only if for every element $t \in \overline{\mathbb{F}_q}$ the Newton polygon of the L -function associated to the exponential sum $\sum \psi(f(x) + tx)$ has a single slope.

Consequently, we have the following:

Theorem 2.1. *If Ai_f does not have finite monodromy (e.g. if $p > 2d-1$), the L -function of the k -th symmetric power of Ai_f is a polynomial:*

$$M_k(f, T) = \det(1 - \text{Frob } T | H_c^1(\mathbb{A}_{\mathbb{F}_q}^1, \text{Sym}^k \text{Ai}_f))$$

While it is tempting to believe that $M_k(f, T)$ is always a polynomial, this is not true, as mentioned in the introduction. In fact, the monodromy group can be finite in certain cases; for instance when $p = 5$ and $f(x) = x^3$, as proven in [19]. In such cases, $H_c^2(\mathbb{A}_{\mathbb{F}_q}^1, \text{Sym}^k \text{Ai}_f)$ will be non-trivial for infinitely many values of k , and consequently $M_k(f, T)$ will have a denominator.

Remark 2.2. Arithmetic difficulties often arise when the characteristic p is small compared to d , as demonstrated above by the link between the finiteness of the monodromy group when $p \leq 2d - 1$ and the Newton polygons of the fibres of the family. By the functional equation, if we denote by $\text{NP}_1(t)$ the slope of the first line segment of the Newton polygon of the fibre t then $\text{NP}_1(t) \leq 1/2$ with equality if and only if the Newton polygon is a single line segment. If $p \equiv 1$ modulo d , and in particular when $p = d + 1$, then by [23, Theorem 3.11] the Newton polygon of every fibre equals the q -adic Newton polygon of the polynomial $\prod_{i=1}^{d-1} (1 - q^{i/d} T)$. Thus, $\text{NP}_1(t) = 1/d$ and so the monodromy group is infinite when $p = d + 1 > 3$.

Let $[f(x)]_{x^N}$ denote the coefficient of x^N in $f(x)$. Suppose $\frac{d}{2} + 1 < p \leq 2d - 1$ and f has coefficients over \mathbb{F}_p . By [22, Theorem 2], if $[(f(x) + tx)^{\lceil \frac{p-1}{d} \rceil}]_{x^{p-1}} \not\equiv 0$ modulo p for some $0 \leq t \leq p - 1$, then $\text{NP}_1(t) \leq \lceil \frac{p-1}{d} \rceil / (p - 1)$ for those t . Hence, the monodromy group is infinite when such a t exists and $d \geq 3$ and $p \geq 7$. Their argument can be extended to show the following. Let $(h(x))_s := h(x)(h(x) - 1) \cdots (h(x) - s + 1)$. Define the linear operator $U : \mathbb{F}_p[x] \rightarrow \mathbb{F}_p$ by sending x^n to 0 if $(p - 1) \nmid n$ and 1 otherwise. Let $c_s := U((f(x) + tx)_s)$. Suppose $c_1 \equiv \cdots \equiv c_{k-1} \equiv 0$ modulo p and $c_k \not\equiv 0 \pmod{p}$ for some t , then $\text{NP}_1(t) \leq \frac{k}{p-1}$. Hence, if this happens for some $k < (p - 1)/2$ then the monodromy group is infinite. For example, for $d > p - 1$ and $f(x) = x^d + x^{p-1}$ then $c_1 = 1$ and hence the monodromy group is infinite for $p \geq 5$.

Lastly, we mention the case when $d = 4$, $p = 7$ and $f \in \mathbb{F}_q[x]$ is not of the form $(x + a)^4 + bx + c$, then by [15, Theorem 4.6] the monodromy of Ai_f is not finite.

3 Computation of the degree of the L -function

We will now study the degree of $M_k(f, T)$ when $p > d$. From the formula above we have

$$\deg(M_k(f, T)) = \dim(H_c^1(\mathbb{A}_{\mathbb{F}_q}^1, \text{Sym}^k \text{Ai}_f)) - \dim(H_c^2(\mathbb{A}_{\mathbb{F}_q}^1, \text{Sym}^k \text{Ai}_f)) = -\chi_c(\mathbb{A}_{\mathbb{F}_q}^1, \text{Sym}^k \text{Ai}_f),$$

where χ_c denotes the Euler characteristic with compact supports. Using the Grothendieck-Néron-Ogg-Shafarevich formula, we have then

$$\begin{aligned} \deg(M_k(f, T)) &= \text{Swan}_{\infty}(\text{Sym}^k \text{Ai}_f) - \text{rank}(\text{Sym}^k \text{Ai}_f) \\ &= \text{Swan}_{\infty}(\text{Sym}^k \text{Ai}_f) - \binom{k + d - 2}{d - 2}. \end{aligned} \tag{2}$$

In order to compute the Swan conductor of $\text{Sym}^k \text{Ai}_f$ we have to study the sheaf Ai_f as a representation of the inertia group I_{∞} of $\mathbb{A}_{\mathbb{F}_q}^1$ at infinity. Since $\mathcal{L}_{\psi(f)}$ is lisse on \mathbb{A}^1 , as a representation of the decomposition group at infinity we have $\text{Ai}_f \cong \mathcal{F}_{\infty, \infty}(\mathcal{L}_{\psi(f)})$, where $\mathcal{F}_{\infty, \infty}$ is the local Fourier transform as defined in [20].

Recently, Fu [9] and, independently, Abbes and Saito [1] have given an explicit description of the different local Fourier transforms for a wide class of ℓ -adic sheaves. We will mainly be using the description given in [1], which works over an arbitrary (not necessarily algebraically closed) perfect base field, and therefore gives an explicit formula for Ai_f as a representation of the decomposition group D_{∞} .

If $S_{(\infty)}$ is the henselization of the local ring of $\mathbb{P}_{\mathbb{F}_q}^1$ at infinity with uniformizer $1/t$, the triple $(\mathcal{L}_{\psi(f(t))}, t, -f'(t))$ is a Legendre triple in the sense of [1, Definition 2.16]. Therefore by [1, Theorem 3.9] we conclude that, as a representation of D_{∞} , Ai_f is isomorphic to

$$(-f')_*(\mathcal{L}_{\psi(f(t))} \otimes \mathcal{L}_{\psi(-tf'(t))} \otimes \mathcal{L}_{\rho(\frac{1}{2}f''(t))} \otimes \mathcal{Q}) = (-f')_*(\mathcal{L}_{\psi(f(t)-tf'(t))} \otimes \mathcal{L}_{\rho(\frac{1}{2}f''(t))} \otimes \mathcal{Q})$$

where ρ is the unique character $I_{\infty} \rightarrow \overline{\mathbb{Q}}_{\ell}^*$ of order 2, \mathcal{L}_{ρ} the corresponding Kummer sheaf and \mathcal{Q} is the pull-back of the character $\text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) \rightarrow \overline{\mathbb{Q}}_{\ell}$ mapping the geometric Frobenius to the quadratic Gauss sum $g(\psi, \rho) := -\sum_{t \in \mathbb{F}_q^*} \psi(t)\rho(t)$.

Write $f(t) = \sum_{i=0}^d c_i t^i$. For simplicity, from now on we will assume that \mathbb{F}_q contains the $2(d - 1)$ -th roots of $-dc_d$ (which can always be achieved by a finite extension of the base field). Following [9, Proposition 3.1] we can find an invertible power series $\sum_{i \geq 0} r_i t^{-i} \in \mathbb{F}_q[[t^{-1}]]$ with $r_0^{d-1} = -dc_d$ such that $u(t) := t \sum_{i \geq 0} r_i t^{-i}$ is a solution to $f'(t) + u(t)^{d-1} = 0$ (the other solutions being $\zeta u(t)$ for every $(d - 1)$ -th root of unity ζ). The map $\phi : 1/t \mapsto 1/u(t)$ defines an automorphism $S_{(\infty)} \rightarrow S_{(\infty)}$, and by construction $-f' = [d - 1] \circ \phi$, where $[d - 1]$ is

the $(d-1)$ -th power map. So Ai_f is isomorphic to

$$\begin{aligned} [d-1]_* \phi_* (\mathcal{L}_{\psi(f(t)-tf'(t))} \otimes \mathcal{L}_{\rho(\frac{1}{2}f''(t))} \otimes \mathcal{Q}) &= [d-1]_* (\phi^{-1})^* (\mathcal{L}_{\psi(f(t)-tf'(t))} \otimes \mathcal{L}_{\rho(\frac{1}{2}f''(t))} \otimes \mathcal{Q}) \\ &= [d-1]_* (\mathcal{L}_{\psi(f(v(t))+v(t)t^{d-1})} \otimes \mathcal{L}_{\rho(\frac{1}{2}f''(v(t)))} \otimes \mathcal{Q}) \\ &= [d-1]_* (\mathcal{L}_{\psi(f(v(t))+v(t)t^{d-1})} \otimes \mathcal{L}_{\rho(\frac{1}{2}f''(v(t)))} \otimes \mathcal{Q}) \end{aligned}$$

since $[d-1]^* \mathcal{Q} = \mathcal{Q}$, where $v(t) := \phi^{-1}(t) = t \sum_{i \geq 0} s_i t^{-i}$.

Let $g(t)$ be the polynomial of degree d obtained from $f(v(t)) + v(t)t^{d-1}$ by removing the terms with negative powers of t . It is important to notice that the coefficients of g are polynomials in the coefficients of f . More precisely, if we write $g(t) = \sum b_i t^i$, the coefficient b_i is a polynomial in the coefficients a_i, a_{i+1}, \dots, a_d of f . Since $\mathcal{L}_{\psi(h(t))}$ is trivial as a representation of D_∞ for any $h(t) \in t^{-1}\mathbb{F}_q[[t^{-1}]]$, we have an isomorphism $\mathcal{L}_{\psi(f(v(t))+v(t)t^{d-1})} \cong \mathcal{L}_{\psi(g(t))}$ as representations of D_∞ .

On the other hand, from $f'(v(t)) + t^{d-1} = 0$ we get $f''(v(t))v'(t) + (d-1)t^{d-2} = 0$, so $\mathcal{L}_{\rho(\frac{1}{2}f''(v(t)))} = \mathcal{L}_{\rho(-\frac{d-1}{2}v'(t)t^{d-2})}$. Since $v'(t) = \sum_{i \geq 0} (1-i)s_i t^{-i} = s_0(1 + \sum_{i \geq 2} (1-i)\frac{s_i}{s_0} t^{-i})$ and $1 + \sum_{i \geq 2} (1-i)\frac{s_i}{s_0} t^{-i}$ is a square in $\mathbb{F}_q[[t^{-1}]]$, we have $\mathcal{L}_{\rho(-\frac{d-1}{2}v'(t)t^{d-2})} = \mathcal{L}_{\rho(-\frac{d-1}{2}s_0 t^{d-2})} = \mathcal{L}_{\rho(\frac{d(d-1)}{2}c_d(s_0 t)^{d-2})}$ (since $s_0^{d-1} = -1/dc_d$). So we finally get

$$\text{Ai}_f \cong [d-1]_* (\mathcal{L}_{\psi(g(t))} \otimes \mathcal{L}_{\rho^d(s_0 t)}) \otimes \mathcal{L}_{\rho(d(d-1)c_d/2)} \otimes \mathcal{Q}. \quad (3)$$

We can now easily compute the Swan conductor at infinity of its symmetric powers. By [17, 1.13.1],

$$\text{Swan}_\infty \text{Sym}^k \text{Ai}_f = \frac{1}{d-1} \text{Swan}_\infty [d-1]^* \text{Sym}^k \text{Ai}_f = \frac{1}{d-1} \text{Swan}_\infty \text{Sym}^k [d-1]^* \text{Ai}_f$$

Lemma 3.1. *Let ζ be a primitive $(d-1)$ -th root of unity in \mathbb{F}_q , I_∞^{d-1} the unique closed subgroup of I_∞ of index $d-1$. As a representation of I_∞^{d-1} , the restriction $[d-1]^* \text{Ai}_f$ of Ai_f is isomorphic to the direct sum*

$$\bigoplus_{i=0}^{d-2} \mathcal{L}_{\psi(g(\zeta^i t))} \otimes \mathcal{L}_{\rho^d(s_0 \zeta^i t)} \cong \bigoplus_{i=0}^{d-2} \mathcal{L}_{\psi(g(\zeta^i t))} \otimes \mathcal{L}_{\rho^d(t)}$$

Proof. Since $(\zeta^i)^* \mathcal{L}_{\psi(g)} = \mathcal{L}_{\psi(g(\zeta^i t))}$, $(\zeta^i)^* \mathcal{L}_{\rho^d(s_0 t)} = \mathcal{L}_{\rho^d(s_0 \zeta^i t)}$ and $[d-1] \circ \zeta^i = [d-1]$ for every i , we have $[d-1]_* (\mathcal{L}_{\psi(g(\zeta^i t))} \otimes \mathcal{L}_{\rho^d(s_0 \zeta^i t)}) = [d-1]_* (\mathcal{L}_{\psi(g(t))} \otimes \mathcal{L}_{\rho^d(s_0 t)})$, and therefore by Frobenius reciprocity $\text{Hom}_{I_\infty^{d-1}}([d-1]^* \text{Ai}_f, \mathcal{L}_{\psi(g(\zeta^i t))} \otimes \mathcal{L}_{\rho^d(s_0 \zeta^i t)}) = \text{Hom}_{I_\infty}(\text{Ai}_f, [d-1]_* (\mathcal{L}_{\psi(g(\zeta^i t))} \otimes \mathcal{L}_{\rho^d(s_0 \zeta^i t)})) = \text{Hom}_{I_\infty}(\text{Ai}_f, \text{Ai}_f) \cong \mathbb{Q}_\ell$ since the latter is an irreducible representation of I_∞ . So for every i , $\mathcal{L}_{\psi(g(\zeta^i t))} \otimes \mathcal{L}_{\rho^d(s_0 \zeta^i t)}$ is a subrepresentation of $[d-1]^* \text{Ai}_f$.

Now $\mathcal{L}_{\psi(g(\zeta^i t))} \otimes \mathcal{L}_{\rho^d(s_0 \zeta^i t)}$ and $\mathcal{L}_{\psi(g(\zeta^j t))} \otimes \mathcal{L}_{\rho^d(s_0 \zeta^j t)}$ are isomorphic if and only if $\mathcal{L}_{\psi(g(\zeta^i t))}$ and $\mathcal{L}_{\psi(g(\zeta^j t))}$ are, if and only if $g(\zeta^i t) - g(\zeta^j t) = h^p - h$ for some $h \in \overline{\mathbb{F}_q}[t]$. Since $p > d$, this can only happen if $g(\zeta^i t) = g(\zeta^j t)$. Comparing the highest degree coefficients we conclude that ζ^i and ζ^j must be equal. Therefore the direct sum of the $\mathcal{L}_{\psi(g(\zeta^i t))} \otimes \mathcal{L}_{\rho^d(s_0 \zeta^i t)}$ for $i = 0, \dots, d-2$ injects into $[d-1]^* \text{Ai}_f$ and we conclude that it must be isomorphic to it, since they have the same rank. \square

Consequently, we have an isomorphism of $\overline{\mathbb{Q}_\ell}[I_\infty]$ -modules

$$\text{Sym}^k [d-1]^* \text{Ai}_f \cong \bigoplus_{a_0 + a_1 + \dots + a_{d-2} = k} \mathcal{L}_{\psi(\sum_{i=0}^{d-2} a_i g(\zeta^i t))} \otimes \mathcal{L}_{\rho^{dk}(t)}.$$

For every finite subset $I \subset \mathbb{Z}$ and every integer $k \geq 0$ define

$$S_{d-1}(k, I) := \{(a_0, \dots, a_{d-2}) \in \mathbb{Z}_{\geq 0}^{d-1} \mid a_0 + a_1 + \dots + a_{d-2} = k, a_0 + a_1 \zeta^i + \dots + a_{d-2} \zeta^{i(d-2)} = 0 \text{ for every } i \in I\}$$

It is clear from the definition that $S_{d-1}(k, I) = S_{d-1}(k, I')$ if $\phi(I) = \phi(I')$, where $\phi : \mathbb{Z} \rightarrow \mathbb{Z}/(d-1)\mathbb{Z}$ is reduction modulo $d-1$. Also, $S_{d-1}(k, I) = \emptyset$ if p does not divide k and $I \cap (d-1)\mathbb{Z} \neq \emptyset$. The number of elements in $S_{d-1}(k, I)$ can be conveniently expressed in terms of a generating function:

Lemma 3.2. *Let $F_{d-1}(I; T) := \sum_{k=0}^\infty \#S_{d-1}(k, I) T^k$. Then*

$$F_{d-1}(I; T) = \frac{1}{q^{\#I}} \sum_{\gamma \in (\mathbb{F}_q)^I} \prod_{j=0}^{d-2} (1 - \psi(\sum_{i \in I} \gamma_i \zeta^{ji}) T)^{-1}$$

where ψ is any non-trivial additive character of \mathbb{F}_q .

Proof. From the definition,

$$F_{d-1}(I; T) = \sum_{(a_0, \dots, a_{d-2}) \in \mathbb{Z}_{\geq 0}^{d-1}} \prod_{i \in I} \delta(a_0 + a_1 \zeta^i + \dots + a_{d-2} \zeta^{i(d-2)}) T^{a_0 + a_1 + \dots + a_{d-2}}$$

where $\delta(a) = 1$ if $a = 0$, 0 otherwise. Equivalently, $\delta(a) = \frac{1}{q} \sum_{\gamma \in \mathbb{F}_q} \psi(\gamma a)$. So we get

$$\begin{aligned} F_{d-1}(I; T) &= \sum_{(a_0, \dots, a_{d-2}) \in \mathbb{Z}_{\geq 0}^{d-1}} \prod_{i \in I} \frac{1}{q} \sum_{\gamma_i \in \mathbb{F}_q} \psi(\gamma_i (a_0 + a_1 \zeta^i + \dots + a_{d-2} \zeta^{i(d-2)})) T^{a_0 + a_1 + \dots + a_{d-2}} \\ &= \sum_{(a_0, \dots, a_{d-2}) \in \mathbb{Z}_{\geq 0}^{d-1}} \sum_{\gamma \in (\mathbb{F}_q)^I} \frac{1}{q^{\#I}} \left(\prod_{i \in I} \psi(\gamma_i a_0) \right) T^{a_0} \left(\prod_{i \in I} \psi(\gamma_i a_1 \zeta^i) \right) T^{a_1} \dots \left(\prod_{i \in I} \psi(\gamma_i a_{d-2} \zeta^{i(d-2)}) \right) T^{a_{d-2}} \\ &= \frac{1}{q^{\#I}} \sum_{\gamma \in (\mathbb{F}_q)^I} \sum_{(a_0, \dots, a_{d-2}) \in \mathbb{Z}_{\geq 0}^{d-1}} \psi\left(\sum \gamma_i\right)^{a_0} T^{a_0} \psi\left(\sum \gamma_i \zeta^i\right)^{a_1} T^{a_1} \dots \psi\left(\sum \gamma_i \zeta^{i(d-2)}\right)^{a_{d-2}} T^{a_{d-2}} \\ &= \frac{1}{q^{\#I}} \sum_{\gamma \in (\mathbb{F}_q)^I} \left(\sum_{a_0 \in \mathbb{Z}_{\geq 0}} \psi\left(\sum \gamma_i\right)^{a_0} T^{a_0} \right) \left(\sum_{a_1 \in \mathbb{Z}_{\geq 0}} \psi\left(\sum \gamma_i \zeta^i\right)^{a_1} T^{a_1} \right) \dots \left(\sum_{a_{d-2} \in \mathbb{Z}_{\geq 0}} \psi\left(\sum \gamma_i \zeta^{i(d-2)}\right)^{a_{d-2}} T^{a_{d-2}} \right) \\ &= \frac{1}{q^{\#I}} \sum_{\gamma \in (\mathbb{F}_q)^I} \prod_{j=0}^{d-2} (1 - \psi\left(\sum \gamma_i \zeta^{j i}\right) T)^{-1}. \end{aligned}$$

□

Write $g(t) = \sum_{j=0}^d b_j t^j$, and let $J = \{1 \leq j \leq d | b_j \neq 0\}$ and $J_{\geq j} := J \cap \{j, j+1, \dots, d\}$ for every $j \in \{1, \dots, d, d+1\}$. We have

$$\begin{aligned} \text{Swan}_{\infty} \text{Sym}^k [d-1]^* \text{Ai}_f &= \sum_{a_0 + a_1 + \dots + a_{d-2} = k} \text{Swan}_{\infty} \mathcal{L}_{\psi(\sum_{i=0}^{d-2} a_i g(\zeta^i t))} \otimes \mathcal{L}_{\rho^{dk}(t)} \\ &= \sum_{a_0 + a_1 + \dots + a_{d-2} = k} \deg \left(\sum_{i=0}^{d-2} a_i g(\zeta^i t) \right) \end{aligned}$$

and

$$\sum_{i=0}^{d-2} a_i g(\zeta^i t) = \sum_{i=0}^{d-2} a_i \sum_{j=0}^d b_j \zeta^{ij} t^j = \sum_{j=0}^d (b_j \sum_{i=0}^{d-2} \zeta^{ij}) t^j$$

so its degree is the greatest j such that $b_j \sum_{i=0}^{d-2} \zeta^{ij} \neq 0$. Therefore we get

$$\begin{aligned} (d-1) \text{Swan}_{\infty} \text{Sym}^k \text{Ai}_f &= \text{Swan}_{\infty} \text{Sym}^k [d-1]^* \text{Ai}_f \\ &= \sum_{j \in J} j \cdot (\#S_{d-1}(k, J_{\geq j+1}) - \#S_{d-1}(k, J_{\geq j})) \\ &= d \binom{k+d-2}{d-2} - \sum_{j \in J} h(j) \cdot \#S_{d-1}(k, J_{\geq j}) \end{aligned}$$

where $h(j) := j - \sup(J - J_{\geq j})$ is the “gap” between the t^j term and the next lower degree term in $g(t)$. Taking the corresponding generating function we get the formula

Corollary 3.3. *Let $G(f; T) := \sum_{k=0}^{\infty} (\text{Swan}_{\infty} \text{Sym}^k \text{Ai}_f) T^k$, then*

$$G(f; T) = \frac{d}{(d-1)(1-T)^{d-1}} - \frac{1}{d-1} \sum_{j \in J} h(j) \cdot F_{d-1}(J_{\geq j}; T)$$

Using the previous formula for the degree, we deduce

Corollary 3.4. *The degree of $M_k(f; T)$ is the k -th coefficient of the power series expansion of*

$$\frac{1}{(d-1)(1-T)^{d-1}} - \frac{1}{d-1} \sum_{j \in J} h(j) \cdot F_{d-1}(J_{\geq j}; T).$$

Corollary 3.5. *For every $J \subset \{1, \dots, d-1\}$, let $\mathcal{P}_d(J)$ be the subspace of the affine space \mathcal{P}_d of polynomials of degree d over k such that $b_j = 0$ if and only if $j \in J$. The sets $\{\mathcal{P}_d(J) | J \subseteq \{1, \dots, d-1\}\}$ define a stratification of \mathcal{P}_d such that the degree of $M_k(f; T)$ is constant in each stratum.*

4 The trivial factor

Suppose $p > d$ and the monodromy of Ai_f is not finite. We will now study the weights of the (reciprocal) roots of the polynomial $M_k(f, T)$. Let us first consider the easier case where d is even, and therefore Ai_f is isomorphic to $[d-1]_* \mathcal{L}_{\psi(g(t))} \otimes \mathcal{L}_{\rho(d(d-1)c_d/2)} \otimes \mathcal{Q}$ as a representation of D_∞ . Let $D_\infty^{d-1} = \text{Gal}(\overline{\mathbb{F}_q}((1/t))/\mathbb{F}_q((1/t^{1/(d-1)})))$, denote by $\alpha : D_\infty^{d-1} \rightarrow \overline{\mathbb{Q}_\ell}^*$ the character corresponding to the sheaf $\mathcal{L}_{\psi(g)}$, and let $b \in I_\infty$ be a generator of the cyclic group $D_\infty/D_\infty^{d-1} \cong I_\infty/I_\infty^{d-1}$. By the explicit description of induced representations, there is a basis $\{v_0, \dots, v_{d-2}\}$ of the underlying vector space V such that $a \cdot v_0 = \alpha(a)v_0$ for every $a \in I_\infty^{d-1}$ and $b \cdot v_i = v_{i+1}$ for $i = 0, \dots, d-3$. Then $b \cdot v_{d-2} = b^{d-1} \cdot v_0 = \alpha(b^{d-1})v_0$. Replacing b by $a^{-1}b$, where $a \in I_\infty^{d-1}$ is an element such that $\alpha(a)^{d-1} = \alpha(b^{d-1})$ (which is always possible since the values of α are the p -th roots of unity and $d-1$ is prime to p since $p > d$) we may assume without loss of generality that $\alpha(b^{d-1}) = 1$.

Furthermore, for any $a \in I_\infty^{d-1}$ we have $a \cdot v_i = (ab^i) \cdot v_0 = (b^i b^{-i} ab^i) \cdot v_0 = \alpha(b^{-i} ab^i)v_i$. So the restriction of Ai_f to D_∞^{d-1} is the direct sum of the characters $a \mapsto \alpha_i(a) := \alpha(b^{-i} ab^i)$. But we already know that it is the direct sum of the characters associated to the sheaves $\mathcal{L}_{\psi(g(\zeta^i t))} \otimes \mathcal{L}_{\rho(d(d-1)c_d/2)} \otimes \mathcal{Q}$, so these two sets of characters are identical. Replacing b by a suitable power of itself we may assume that α_i is the character associated to $\mathcal{L}_{\psi(g(\zeta^i t))} \otimes \mathcal{L}_{\rho(d(d-1)c_d/2)} \otimes \mathcal{Q}$. In particular, $\prod_{i=0}^{d-2} \alpha_i^{a_i}$ is geometrically trivial (that is, trivial on I_∞^{d-1}) if and only if $\sum a_i g(\zeta^i t)$ is a constant in $\mathbb{F}_q[t]$, that is, if and only if $\sum a_i \zeta^{ij} = 0$ for every $j \in J$.

We turn now to the case d odd. Let χ be a multiplicative character of \mathbb{F}_q of order $2(d-1)$ (which exists, since we are assuming that \mathbb{F}_q contains the $2(d-1)$ -th roots of unity). Then by the projection formula Ai_f is isomorphic to $[d-1]_* (\mathcal{L}_{\psi(g(t))} \otimes \mathcal{L}_{\rho(s_0 t)}) \otimes \mathcal{L}_{\rho(d(d-1)c_d/2)} \otimes \mathcal{Q} \cong ([d-1]_* \mathcal{L}_{\psi(g(t))}) \otimes \mathcal{L}_{\chi(s_0 t)} \otimes \mathcal{L}_{\rho(d(d-1)c_d/2)} \otimes \mathcal{Q}$. Let $\alpha_i : D_\infty^{d-1} \rightarrow \overline{\mathbb{Q}_\ell}^*$ (respectively $\beta : D_\infty \rightarrow \overline{\mathbb{Q}_\ell}^*$) be the character corresponding to the sheaf $\mathcal{L}_{\psi(g(\zeta^i t))}$ (resp. $\mathcal{L}_{\chi(s_0 t)}$). Proceeding as in the d even case, we find a generator $b \in I_\infty$ of D_∞/D_∞^{d-1} and a basis $\{v_0, \dots, v_{d-2}\}$ of V such that $a \cdot v_i = \alpha_i(a)\beta(a)v_i$ for $a \in D_\infty^{d-1}$ and $b \cdot v_i = \beta(b)v_{i+1}$ for $i = 0, \dots, d-3$, $b \cdot v_{d-2} = \beta(b)v_0$. In this case, $\prod_{i=0}^{d-2} \alpha_i^{a_i} \beta^{a_i}$ is trivial on I_∞^{d-1} if and only if $\sum a_i g(\zeta^i t)$ is a constant in $\mathbb{F}_q[t]$ and $\sum a_i$ is even (since α_i has order p and β restricted to I_∞^{d-1} has order 2).

We can now compute the dimension of the invariant subspace of the action of I_∞ on $\text{Sym}^k \text{Ai}_f$, in very much the same way it is done for the Kloosterman sheaf in [10, Lemma 2.1]. Its underlying vector space is $\text{Sym}^k V$. An element w is given by a linear combination

$$w = \sum_{a_0 + \dots + a_{d-2} = k} c_{a_0 \dots a_{d-2}} v_0^{a_0} \dots v_{d-2}^{a_{d-2}}.$$

In the d even case we have

$$a \cdot \sum_{a_0 + \dots + a_{d-2} = k} c_{a_0 \dots a_{d-2}} v_0^{a_0} \dots v_{d-2}^{a_{d-2}} = \sum_{a_0 + \dots + a_{d-2} = k} c_{a_0 \dots a_{d-2}} (\alpha_0^{a_0} \dots \alpha_{d-2}^{a_{d-2}})(a) v_0^{a_0} \dots v_{d-2}^{a_{d-2}}$$

for $a \in I_\infty^{d-1}$ and

$$b \cdot \sum_{a_0 + \dots + a_{d-2} = k} c_{a_0 \dots a_{d-2}} v_0^{a_0} \dots v_{d-2}^{a_{d-2}} = \sum_{a_0 + \dots + a_{d-2} = k} c_{a_0 \dots a_{d-2}} v_1^{a_0} v_2^{a_1} \dots v_0^{a_{d-2}}.$$

So w is fixed by I_∞ if and only if the character $\alpha_0^{a_0} \dots \alpha_{d-2}^{a_{d-2}}$ is trivial whenever $c_{a_0 \dots a_{d-2}} \neq 0$ and $c_{a_0 \dots a_{d-2}} = c_{a_{d-2} a_0 \dots a_{d-3}}$ for all a_0, \dots, a_{d-2} . A basis for the invariant subspace is thus given by all distinct sums of the form (setting $v_{d-1+l} := v_l$ for all $l \geq 0$):

$$\sum_{j=0}^{d-2} v_j^{a_0} v_{j+1}^{a_1} \dots v_{j+d-2}^{a_{d-2}}$$

for all a_0, \dots, a_{d-2} such that $\alpha_0^{a_0} \dots \alpha_{d-2}^{a_{d-2}}$ is trivial, that is, such that $\sum a_i \zeta^{ij} = 0$ in \mathbb{F}_q for every $j \in J$.

In the d odd case we get

$$g \cdot \sum_{a_0 + \dots + a_{d-2} = k} c_{a_0 \dots a_{d-2}} v_0^{a_0} \dots v_{d-2}^{a_{d-2}} = \sum_{a_0 + \dots + a_{d-2} = k} c_{a_0 \dots a_{d-2}} (\alpha_0^{a_0} \dots \alpha_{d-2}^{a_{d-2}})(g) \beta^k(g) v_0^{a_0} \dots v_{d-2}^{a_{d-2}}$$

for $g \in I_\infty^{d-1}$ and

$$h \cdot \sum_{a_0 + \dots + a_{d-2} = k} c_{a_0 \dots a_{d-2}} v_0^{a_0} \dots v_{d-2}^{a_{d-2}} = \sum_{a_0 + \dots + a_{d-2} = k} c_{a_0 \dots a_{d-2}} \beta(h)^k v_1^{a_0} v_2^{a_1} \dots v_0^{a_{d-2}}.$$

So w is fixed by I_∞ if and only if the character $\alpha_0^{a_0} \dots \alpha_{d-2}^{a_{d-2}} \beta^k$ of I_∞^{d-1} is trivial whenever $c_{a_0 \dots a_{d-2}} \neq 0$ and $c_{a_0 \dots a_{d-2}} = c_{a_{d-2} a_0 \dots a_{d-3}} \beta(h)^k$ for all a_0, \dots, a_{d-2} . Since all α_i 's have order p and the restriction of β to I_∞^{d-1} has

order 2, $\alpha_0^{a_0} \cdots \alpha_{d-2}^{a_{d-2}} \beta^k$ is trivial if and only if both $\alpha_0^{a_0} \cdots \alpha_{d-2}^{a_{d-2}}$ and β^k are trivial as characters of I_∞^{d-1} , that is, if and only if $\sum a_i \zeta^{ij} = 0$ in \mathbb{F}_q for every $j \in J$ and k is even. In particular, there are no non-zero invariants for I_∞ if k is odd. If k is even, a generating set for the invariant subspace is given by all distinct sums of the form

$$\sum_{j=0}^{d-2} \beta(h)^{jk} v_j^{a_0} v_{j+1}^{a_1} \cdots v_{j+d-2}^{a_{d-2}}$$

for all a_0, \dots, a_{d-2} such that $\sum a_i \zeta^{ij} = 0$ in \mathbb{F}_q for every $j \in J$. Let r be the size of the orbit of (a_0, \dots, a_{d-2}) under the action of $\mathbb{Z}/(d-1)\mathbb{Z}$ by cyclic permutations. If $r \neq d-1$, we can write

$$\sum_{j=0}^{d-2} \beta(h)^{jk} v_j^{a_0} v_{j+1}^{a_1} \cdots v_{j+d-2}^{a_{d-2}} = \sum_{j=0}^{r-1} \beta(h)^{jk} (1 + \beta(h)^{rk} + \cdots + \beta(h)^{(\frac{d-1}{r}-1)rk}) v_j^{a_0} v_{j+1}^{a_1} \cdots v_{j+d-2}^{a_{d-2}}.$$

Notice that k must be a multiple of $\frac{d-1}{r}$, since $k = \sum_{i=0}^{d-2} a_i = \frac{d-1}{r} \sum_{i=0}^{r-1} a_i$. If $\frac{rk}{d-1}$ is odd we have

$$1 + \beta(h)^{rk} + \cdots + \beta(h)^{(\frac{d-1}{r}-1)rk} = \frac{1 - \beta(h)^{(d-1)k}}{1 - \beta(h)^{rk}} = 0,$$

so the above sum vanishes. On the other hand, if $\frac{rk}{d-1}$ is even it is clear that the element

$$\sum_{j=0}^{d-2} \beta(h)^{jk} v_j^{a_0} v_{j+1}^{a_1} \cdots v_{j+d-2}^{a_{d-2}} = \frac{d-1}{r} \sum_{j=0}^{r-1} \beta(h)^{jk} v_j^{a_0} v_{j+1}^{a_1} \cdots v_{j+d-2}^{a_{d-2}}$$

is non-zero, and to different orbits correspond different elements. To summarize, we have

Proposition 4.1. *Let $T_{d-1}(k, J)$ be the set of orbits of the action of $\mathbb{Z}/(d-1)\mathbb{Z}$ on the set $S_{d-1}(k, J)$ by cyclic permutations, and let $U_{d-1}(k, J)$ be the subset of orbits such that $\frac{rk}{d-1}$ is even, where r is their cardinality. If d is even, the invariant subspace of the representation $\text{Sym}^k \text{Ai}_f$ of I_∞ has dimension $\#T_{d-1}(k, J)$. If d is odd and k is even, it has dimension $\#U_{d-1}(k, J)$. If d and k are odd, the representation has no non-zero invariants.*

The sequences $\#T_{d-1}(k, J)$ and $\#U_{d-1}(k, J)$ can also be described by means of generating functions. By Burnside's lemma, the dimension of the invariant subspace for d even is given by

$$\#T_{d-1}(k, J) = \frac{1}{d-1} \sum_{r=1}^{d-1} \#\{(a_0, a_1, \dots, a_{d-2}) | a_i = a_{i+r \bmod d-1}\} = \frac{1}{d-1} \sum_{r|d-1} \phi\left(\frac{d-1}{r}\right) \#S_r\left(\frac{kr}{d-1}, J\right)$$

where $S_r(k, J) = \emptyset$ if k is not an integer and ϕ is Euler's totient function. So the generating function for the sequence $\{\#T_{d-1}(k, J) | k \geq 0\}$ is

$$\begin{aligned} G_{d-1}(J; T) &:= \sum_{k=0}^{\infty} \#T_{d-1}(k, J) T^k \\ &= \sum_{k=0}^{\infty} \frac{1}{d-1} T^k \sum_{r|d-1} \phi\left(\frac{d-1}{r}\right) \#S_r\left(\frac{kr}{d-1}, J\right) \\ &= \frac{1}{d-1} \sum_{r|d-1} \phi\left(\frac{d-1}{r}\right) \sum_{\frac{d-1}{r} | k} \#S_r\left(\frac{kr}{d-1}, J\right) T^k \\ &= \frac{1}{d-1} \sum_{r|d-1} \phi\left(\frac{d-1}{r}\right) \sum_{s=0}^{\infty} \#S_r(s, J) T^{\frac{d-1}{r}s} \\ &= \frac{1}{d-1} \sum_{r|d-1} \phi\left(\frac{d-1}{r}\right) F_r(J; T^{\frac{d-1}{r}}) \end{aligned}$$

Next, suppose that d is odd, and let $(a_0, \dots, a_{d-2}) \in S_{d-1}(k, J)$. Let r be the number of elements in its orbit. Then $\sum_{i=0}^{r-1} a_i = \frac{kr}{d-1}$. We want to count the number of orbits such that this value is even. Since $k = \frac{kr}{d-1} \cdot \frac{d-1}{r}$, if the largest power of 2 that divides $d-1$ is smaller than the largest power of 2 dividing k , $\frac{kr}{d-1}$ must always be even. Suppose that the largest power of 2 that divides k , $2^{\alpha(k)}$, divides $d-1$. Then $\frac{kr}{d-1}$ is odd if and only if $2^{\alpha(k)}$

divides $\frac{d-1}{r}$, if and only if r divides $\frac{d-1}{2^{\alpha(k)}}$. Therefore $\#U_{d-1}(k, J) = \#T_{d-1}(k, J)$ if $2^{\alpha(k)}$ does not divide $d-1$ and $\#T_{d-1}(k, J) - \#T_{\frac{d-1}{2^{\alpha(k)}}}(\frac{k}{2^{\alpha(k)}}, J)$ if it does. The generating function is then

$$\begin{aligned} \sum_{k=0}^{\infty} \#U_{d-1}(k, J) T^k &= \sum_{k=0}^{\infty} \#T_{d-1}(k, J) T^k - \sum_{j \geq 1; 2^j | d-1} \sum_{l \text{ odd}} \#T_{\frac{d-1}{2^j}}(l, J) T^{2^j l} \\ &= G_{d-1}(J; T) - \sum_{j \geq 1; 2^j | d-1} H_{\frac{d-1}{2^j}}(J; T^{2^j}) \end{aligned}$$

where

$$H_r(J; T) := \frac{1}{2} (G_r(J; T) - G_r(J; -T)).$$

Let $F \in D_{\infty}^{d-1} \subset D_{\infty}$ be a geometric Frobenius element, and $w = \sum_{j=0}^{d-2} v_j^{a_0} v_{j+1}^{a_1} \cdots v_{j+d-2}^{a_{d-2}}$ (resp. $w = \sum_{j=0}^{d-2} \beta(h)^{jk} v_j^{a_0} v_{j+1}^{a_1} \cdots v_{j+d-2}^{a_{d-2}}$) a generator of the I_{∞} -invariant subspace of $\text{Sym}^k V$. F acts on $v_j^{a_0} v_{j+1}^{a_1} \cdots v_{j+d-2}^{a_{d-2}}$ via the character corresponding to $\mathcal{L}_{\psi(\sum a_i g(\zeta^{j+i} t))} \otimes \mathcal{L}_{\rho(d(d-1)c_d/2)}^{\otimes k} \otimes \mathcal{Q}^{\otimes k}$ (resp. $\mathcal{L}_{\psi(\sum a_i g(\zeta^{j+i} t))} \otimes \mathcal{L}_{\rho(\prod (s_0 \zeta^{j+i} t)^{a_i})} \otimes \mathcal{L}_{\rho(d(d-1)c_d/2)}^{\otimes k} \otimes \mathcal{Q}^{\otimes k}$). Since $\sum a_i g(\zeta^{j+i} t)$ must be a constant polynomial, we have $\mathcal{L}_{\psi(\sum a_i g(\zeta^{j+i} t))} \cong \mathcal{L}_{\psi(kb_0)}$. Additionally, if d is odd and k even, $\mathcal{L}_{\rho(\prod (s_0 t)^{a_i})} = \mathcal{L}_{\rho(s_0 t)^k}$ is trivial. We conclude:

Proposition 4.2. *A Frobenius geometric element at infinity acts on the I_{∞} -invariant subspace of $\text{Sym}^k \text{Ai}_f$ by multiplication by $\psi(kb_0) \rho(d(d-1)c_d/2)^k g(\psi, \rho)^k$.*

As an immediate consequence we get

Corollary 4.3. *The local L -function of $\text{Sym}^k \text{Ai}_f$ at infinity $\det(1 - \text{Frob } T | (\text{Sym}^k \text{Ai}_f)^{I_{\infty}})$ is given by $(1 - \psi(kb_0) \rho(d(d-1)c_d/2)^k g(\psi, \rho)^k T)^{\#T_{d-1}(k, J)}$ if d is even, $(1 - \psi(kb_0) \rho(d(d-1)c_d/2)^k g(\psi, \rho)^k T)^{\#U_{d-1}(k, J)}$ if d is odd and k is even, and 1 if d and k are odd.*

Theorem 4.4. *The polynomial $M_k(f, T)$ decomposes as a product $P_k(f, T) Q_k(f, T)$, where $Q_k(f, T)$ is given by the formula in Corollary 4.3 and $P_k(d, T)$ satisfies a functional equation*

$$P(T) = c T^r \overline{P(1/q^{k+1} T)}$$

where $|c| = q^{r(k+1)/2}$ and r is its degree.

Proof. Let $j : \mathbb{A}^1 \rightarrow \mathbb{P}^1$ be the inclusion. From the exact sequence

$$0 \rightarrow \text{Sym}^k \text{Ai}_f \rightarrow j_{\star} \text{Sym}^k \text{Ai}_f \rightarrow (j_{\star} \text{Sym}^k \text{Ai}_f)_{\infty} \rightarrow 0$$

we get an exact sequence of $\text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ -modules

$$0 \rightarrow (j_{\star} \text{Sym}^k \text{Ai}_f)^{I_{\infty}} \rightarrow H_c^1(\mathbb{A}^1, \text{Sym}^k \text{Ai}_f) \rightarrow H^1(\mathbb{P}^1, j_{\star} \text{Sym}^k \text{Ai}_f) \rightarrow 0$$

and therefore a decomposition

$$\begin{aligned} M_k(f, T) &= \det(1 - \text{Frob } T | H_c^1(\mathbb{A}^1, \text{Sym}^k \text{Ai}_f)) \\ &= \det(1 - \text{Frob } T | (j_{\star} \text{Sym}^k \text{Ai}_f)^{I_{\infty}}) \det(1 - \text{Frob } T | H^1(\mathbb{P}^1, j_{\star} \text{Sym}^k \text{Ai}_f)). \end{aligned}$$

The first factor is described by the previous corollary. On the other hand, by [7, Théorème 1.3] we have a perfect pairing

$$H^1(\mathbb{P}^1, j_{\star} \text{Sym}^k \text{Ai}_f) \times H^1(\mathbb{P}^1, j_{\star} \widehat{\text{Sym}^k \text{Ai}_f}) \rightarrow \overline{\mathbb{Q}}_{\ell}(-k-1)$$

where $\widehat{\text{Ai}_f}$ is the dual of Ai_f , which is constructed in the same way as Ai_f using the complex conjugate character $\bar{\psi}$ instead of ψ . If the eigenvalues of the action of Frobenius on $H^1(\mathbb{P}^1, j_{\star} \text{Sym}^k \text{Ai}_f)$ are $\alpha_1, \dots, \alpha_r$, so that $P_k(f, T) = \prod (1 - \alpha_i T)$, it follows that $\overline{P_k(f, T)} = \prod (1 - (q^{k+1}/\alpha_i) T)$ and therefore the functional equation holds. Applying the functional equation twice we get $|c| = q^{r(k+1)/2}$. \square

5 Some special cases

We will now see how the previous results apply to some special values of f . First, consider the case $f(t) = t^d$. In this case the equation $f'(t) + u(t)t^{d-1} = 0$ gives $u(t) = r_0 t$, where $r_0^{d-1} = -d$. Then $v(t) = t/r_0$, and $g(t) = f(v(t)) + v(t)t^{d-1} = t^d(1/r_0^d + 1/r_0) = \frac{d-1}{dr_0}t^d$. By corollary 3.4, we get that the degree of $M_k(f; T)$ is the k -th coefficient in the power series expansion of

$$\frac{1}{d-1} \left(\frac{1}{(1-T)^{d-1}} - dF_{d-1}(\{1\}; T) \right)$$

where

$$F_{d-1}(\{1\}; T) = \frac{1}{q} \sum_{\gamma \in \mathbb{F}_q} \prod_{j=0}^{d-2} (1 - \psi(\gamma \zeta^j) T)^{-1}.$$

Explicitly,

$$\deg M_k(f, T) = \frac{1}{d-1} \left(\binom{k+d-2}{d-2} - d \cdot \#S_{d-1}(k, \{1\}) \right).$$

In particular, for $d = 3$

$$F_2(\{1\}; T) = \frac{1}{q} \sum_{\gamma \in \mathbb{F}_q} (1 - \psi(\gamma)T)^{-1} (1 - \psi(-\gamma)T)^{-1} = \frac{1}{p} \sum_{m=0}^{p-1} (1 - \exp(\frac{2\pi i m}{p})T)^{-1} (1 - \exp(\frac{-2\pi i m}{p})T)^{-1}.$$

It is easily checked that $S_2(k, \{1\}) := \{(a, b) | a + b = k, a \equiv b \pmod{p}\}$ has $\lfloor \frac{k}{p} \rfloor + \delta$ elements, where $\delta = 0$ (resp. $\delta = 1$) if $k - \lfloor \frac{k}{p} \rfloor$ is odd (resp. even). So in this case we get an explicit formula for the degree:

$$\deg M_k(f(t) = t^3; T) = \frac{1}{2} \left(k + 1 - 3 \left(\left\lfloor \frac{k}{p} \right\rfloor + \delta \right) \right)$$

If $p > k$ this gives $(k+1)/2$ for k odd and $(k-2)/2$ for k even.

Corollary 4.3 states for $f(t) = t^d$ that the local L -function of $\text{Sym}^k \text{Ai}_f$ at infinity is $(1 - \rho(d(d-1)/2)^k g(\psi, \rho)^k T)^{\#T_{d-1}(k, J)}$ if d is even, $(1 - \rho(d(d-1)/2)^k g(\psi, \rho)^k T)^{\#U_{d-1}(k, J)}$ if d is odd and k is even and 1 if d and k are odd. For $d = 3$, we can again provide a more explicit expression.

Since 3 is odd, the local L -function is 1 for k odd. For k even, we can write $\#S_2(k, \{1\}) = \lfloor \frac{k}{p} \rfloor + \delta = 2\lfloor \frac{k}{2p} \rfloor + 1$. Every orbit of $\mathbb{Z}/2\mathbb{Z}$ acting on $S_2(k, \{1\})$ has two elements except for $\{(k/2, k/2)\}$, so $\#T_2(k, \{1\}) = \lfloor \frac{k}{2p} \rfloor + 1$. $U_2(k, \{1\})$ contains the orbits such that rk is a multiple of 4. If $k \equiv 0 \pmod{4}$ this includes all orbits. If $k \equiv 2 \pmod{4}$ the orbit $\{(\frac{k}{2}, \frac{k}{2})\}$ must be excluded. So the trivial factor for k even is

$$\begin{aligned} (1 - g(\psi, \rho)^k T)^{\lfloor \frac{k}{2p} \rfloor} & \quad \text{for } k \equiv 2 \pmod{4} \\ (1 - g(\psi, \rho)^k T)^{\lfloor \frac{k}{2p} \rfloor + 1} & \quad \text{for } k \equiv 0 \pmod{4} \end{aligned}$$

In particular, for $p > \frac{k}{2}$ the trivial factor of $M_k(t^3, T)$ is 1 if $k \equiv 2 \pmod{4}$ and $(1 - g(\psi, \rho)^k T)$ if $k \equiv 0 \pmod{4}$.

We will now consider the case where $g(t) = \sum b_i t^i$ has $b_i \neq 0$ for $i = 1, \dots, d-2$. This includes the generic case where all coefficients of $g(t)$ are non-zero as a special case. Suppose first that $b_{d-1} = 0$ (or, equivalently, that $c_{d-1} = 0$). $S_{d-1}(k, J)$ is the set of all $(a_0, \dots, a_{d-2}) \in \mathbb{Z}_{\geq 0}^{d-1}$ such that $\sum a_i = k$ and $\sum a_i \zeta^{ji} = 0$ for all $j = 1, \dots, d-2$. The system of equations $\{\sum_i \zeta^{ij} x_i = 0 | j = 1, \dots, d-2\}$ has rank $d-2$ (since the $(d-2) \times (d-2)$ minors are Vandermonde determinants) and has $(1, 1, \dots, 1)$ as a solution, so all solutions must be of the form (a, a, \dots, a) modulo p for some a . Therefore

$$\begin{aligned} F_{d-1}(J; T) &:= \sum_{k=0}^{\infty} \#S_{d-1}(k, J) T^k \\ &= \sum_{r=0}^{p-1} \sum_{a_0, \dots, a_{d-2}=0}^{\infty} T^{(r+s_0 p) + \dots + (r+s_{d-2} p)} \\ &= \sum_{r=0}^{p-1} T^{(d-1)r} \sum_{a_0, \dots, a_{d-2}=0}^{\infty} T^{p(a_0 + \dots + a_{d-2})} \\ &= \frac{1 - T^{(d-1)p}}{(1 - T^p)^{d-1} (1 - T^{d-1})} \end{aligned}$$

Suppose now that $b_{d-1} \neq 0$ (or, equivalently, that $c_{d-1} \neq 0$). Making the change of variable $\hat{f}(t) = f(t - \frac{c_{n-1}}{nc_n})$ we eliminate the degree $d-1$ term. Moreover, $\text{Ai}_{\hat{f}} = R^1\pi_{t!}\mathcal{L}_{\psi(f(x - \frac{c_{n-1}}{nc_n}) + tx)} = R^1\pi_{t!}\mathcal{L}_{\psi(f(x) + t(x + \frac{c_{n-1}}{nc_n}))} = \text{Ai}_f \otimes \mathcal{L}_{\psi(\frac{c_{n-1}}{nc_n}t)}$ and thus $\text{Sym}^k \text{Ai}_f = (\text{Sym}^k \text{Ai}_{\hat{f}}) \otimes \mathcal{L}_{\psi(-\frac{c_{n-1}}{nc_n}t)}^{\otimes k}$. As a representation of D_{∞} , we have then $\text{Ai}_f = [d-1]_{\star}(\mathcal{L}_{\psi(\hat{g}(t))} \otimes \mathcal{L}_{\rho^d(s_0t)}) \otimes \mathcal{L}_{\rho(d(d-1)c_d/2)} \otimes \mathcal{Q} \otimes \mathcal{L}_{\psi(-\frac{c_{n-1}}{nc_n}t)} = [d-1]_{\star}(\mathcal{L}_{\psi(\hat{g}(t) - \frac{c_{n-1}}{nc_n}t^{d-1})} \otimes \mathcal{L}_{\rho^d(s_0t)}) \otimes \mathcal{L}_{\rho(d(d-1)c_d/2)} \otimes \mathcal{Q}$. In other words, $g(t) = \hat{g}(t) - \frac{c_{n-1}}{nc_n}t^{d-1}$.

If p divides k , the condition $\sum_i a_i \zeta^{ij}$ for $j = d-1$ is void, so both the dimension of $M_k(f; T)$ and the trivial factor at infinity behave as in the $b_{d-1} = 0$ case. If p does not divide k , the condition $\sum_i a_i \zeta^{ij}$ does never hold for $j = d-1$, so $S_{d-1}(k, J_{\geq j}) = \emptyset$ for $j = 1, \dots, d-1$. In particular, the trivial factor of $M_k(f; T)$ is 1. Furthermore, applying the formula for the degree, we get

$$\deg M_k(f, T) = \frac{1}{d-1} \left(\binom{k+d-2}{d-2} - \#S_{d-1}(k, \{1\}) \right).$$

As a final example, suppose that $d-1$ is prime and p is a multiplicative generator of \mathbb{F}_{d-1} . In this case, all non-trivial $(d-1)$ -th roots of unity are conjugate over \mathbb{F}_p , so $a_0 + a_1\zeta + \dots + a_{d-2}\zeta^{d-2} = 0$ if and only if $a_0 + a_1\zeta^j + \dots + a_{d-2}\zeta^{(d-2)j} = 0$ for any $j = 1, 2, \dots, d-2$. Therefore $S_{d-1}(k, \{1\}) = S_{d-1}(k, J)$ for every $J \subset \mathbb{Z}$ such that $J \cap (d-1)\mathbb{Z} = \emptyset$. As in the previous example, we conclude that, if $c_{d-1} = 0$,

$$F_{d-1}(J_{\geq j}; T) = \frac{1 - T^{(d-1)p}}{(1 - Tp)^{d-1}(1 - T^{d-1})}$$

for every $j \in J$. By corollary 3.4, the degree of $M_k(f; T)$ is the k -th coefficient of the power series expansion of

$$\frac{1}{(d-1)(1-T)^{d-1}} - \frac{1}{d-1} \cdot \frac{1 - T^{(d-1)p}}{(1 - Tp)^{d-1}(1 - T^{d-1})} \sum_{j \in J} h(j) = \frac{1}{(d-1)(1-T)^{d-1}} - \frac{d}{d-1} \cdot \frac{1 - T^{(d-1)p}}{(1 - Tp)^{d-1}(1 - T^{d-1})}.$$

If $c_{d-1} \neq 0$ we have, as in the previous example, the same formula for the degree if k is a multiple of p , and the k -th coefficient in the power series expansion of

$$\frac{1}{(d-1)(1-T)^{d-1}} - \frac{1}{d-1} \cdot \frac{1 - T^{(d-1)p}}{(1 - Tp)^{d-1}(1 - T^{d-1})}$$

if k is prime to p .

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